Homogenization of Maxwell's equations in bianisotropic materials

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Let Ω be a domain in $\mathbb{R}^3$ and $\partial\Omega$ Lipschitz. We consider the typical Maxwell problem with equations

$$\frac{\partial}{\partial t} D(x,t) = \operatorname{curl} H(x,t) + F(x,t) \tag{1}$$
$$\frac{\partial}{\partial t} B(x,t) = -\operatorname{curl} E(x,t) + G(x,t) \tag{2}$$

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$$D(x,t) = \eta E + \xi H + \eta_d \star E + \xi_d \star H \tag{3}$$
$$B(x,t) = \zeta E + \mu H + \zeta_d \star E + \mu_d \star H \tag{4}$$

where η, ξ, ζ and μ are 3×3 matrices. Likely η_d, ξ_d, ζ_d and μ_d are also 3×3 matrices.

Let $u := (E, H)^T$, $J := (F, G)^T$, $d := (D, B)^T$,
$u^0(x) := (E^0(x), H^0(x))^T$,

$$A(x) := \begin{pmatrix} \eta & \xi \\ \zeta & \mu \end{pmatrix}, G_d(x,t) := \begin{pmatrix} \eta_d & \xi_d \\ \zeta_d & \mu_d \end{pmatrix}$$

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and $M := \begin{pmatrix} 0 & \operatorname{curl} \\ -\operatorname{curl} & 0 \end{pmatrix}$ then the system becomes $(P1)$

$$\begin{cases} \frac{\partial}{\partial t}(A(x)u(x,t) + (G_d \star u)(x,t)) = Mu(x,t) + J(x,t) \\ u(x,0) = u^0(x) \\ \hat{\eta} \times u(x,t) = 0. \end{cases} \quad (5)$$

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We assume that all the fields which are functions of the spatial variable x and the time variable t are considered to be functions of the time variable t in a suitable Banach space.

We also assume that matrix A is symmetric and coercive i.e

$$x^T A(x) x \geq \beta |x|^2, \text{ for any } x \in \mathbb{R}^6.$$

Theorem 1

Let $A \in L^\infty(\Omega; \mathbb{R}^{36})$ *and* $G_d \in W^{2,1}(0, T; L^\infty(\Omega; \mathbb{R}^{36}))$ *be* 6×6 *matrices,* $u^0 \in X_M := H_0(\text{curl}, \Omega) \times H(\text{curl}, \Omega)$ *and* $J \in W^{1,1}(0, T; L^2(\Omega; \mathbb{R}^6))$ *be 6-vectors then the problem* $(P1)$ *has unique solution*

$$u \in W^{1,\infty}(0, T; L^2(\Omega; \mathbb{R}^6)) \cap L^\infty(0, T; X_m)$$

which satisfies the estimate

$$\|u\|_{L^\infty(0,T;X_M)} + \|\frac{du}{dt}\|_{L^\infty(0,T;L^2(\Omega;\mathbb{R}^6))} \leq c(\|J\|_{W^{1,1}(0,T;L^2(\Omega;\mathbb{R}^6))} + \|u^0\|_{X_M}) \tag{6}$$

where c *is a positive constant which depends on* $\|A\|_{L^\infty}$ *and* $\|G_d\|_{L^\infty}$*.*

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We obtain the existence of the solution using the following lemma:

Lemma 2

If A, R are $m \times m$ matrices with matrix A symmetric and coercive, $K \in W^{r,1}(0,T;\mathbb{R}^{m^2})$ and $B \in W^{r,1}(0,T;\mathbb{R}^m)$,$r = 1, 2$ then the integral equation Voltera

$$AU(t) + \int_0^t (K(t-s) - R)U(s)ds = B(t), \;\; t \in [0,T]$$

has a unique solution $U(t) \in W^{r,1}(0,T,\mathbb{R}^m)$.

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In order to proof the lemma we need the Fredholm theory.

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Looking for a solution of the form

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Step 2: Estimates Fisrtly, we prove the following equality

$$\int_\Omega \frac{d}{dt}(A(x)u_m(t))u_m(t)dx + \int_\Omega \frac{d}{dt}(G_d(t) \star u_m(t))u_m(t) = \int_\Omega J(t)u_m(t)dx \tag{7}$$

which results from the main equation of problem $P1$ and the relations $H \operatorname{curl} E - E \operatorname{curl} H = \operatorname{div}(E \times H)$, $\hat{\eta}(E \times H) = H(\hat{\eta} \times E)$ and the perfect boudary condition $\hat{\eta} \times E$.

Step 3: We prove the equality

$$(A(x)u_m(t), u_m(t)) = -2\int_0^t (\dot{G}(s) \star u_m(s), u_m(s))ds - 2\int_0^t (G_d(0)u_m(s), u_m(s))ds + (A(x)u_m(0), u_m(0)) + 2\int_0^t (J(s), u_m(s))ds. \quad (8)$$

We obtain the above relation by using some other equlities and after some suitable integrations.

Step 4: We estimate each term of the equation (8) and by using the coercivity of A, the Cauchy-Schwartz inequality , the theory of norms and the relation $2ab \leq \epsilon a^2 + \frac{1}{\epsilon} b^2$ for $\epsilon > 0$ we deduce that

$$v_m^2(t) \leq \frac{2}{\beta} \|A\|_{L^\infty} \|u^0\|^2 + \frac{4}{\beta} \|J\|_{L^2} + \frac{2}{\beta} \int_0^t v_m^2(s) \theta(s) ds \qquad (9)$$

where β is a constant, $\theta(s) := 2(\int_0^s \|\dot{G}(\sigma)\|_{L^\infty} d\sigma + \|G_d(0)\|_{L^\infty})$ and $v_m(s) := sup_{0 \leq r \leq s} \|u_m(r)\|_{L^2}$.

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$$\|u_m(t)\|_{L^2} \leq c \sqrt{\|u_0\|_{L^2}^2 + \|J\|_{L^1}^2} \leq c(\|u_0\|_{L^2}^2 + \|J\|_{L^1}^2)$$

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and we obtain the estimate

$$\|u_m\|_{L^\infty}(0, T, L^2(\Omega, R^6)) \leq \{\|u_0\|_{X_M} + \|J\|_{L^1(0,T,L^2(\Omega,R^6))}\}.$$

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$$u_m \rightharpoonup u \text{ in } L^\infty(0, T, L^2(\Omega, \mathbb{R}^6))$$

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and as a result $u \in W^{1,\infty}(0, T, L^2(\Omega, \mathbb{R}^6))$. Now, from $\frac{d}{dt}(Au_m + G_d \star u_m) = Mu_m + j$ we obtain $\frac{d}{dt}\int_\Omega (Au_m + G_d \star u_m)e_i dx = -\int_\Omega Mu_m e_i dx + \int_\Omega je_i dx$.Taking into consideration the uniqueness of the weak* limit, the density of V_m in $\mathcal{X}_\mathcal{M}$ and the density of $\mathcal{X}_\mathcal{M}$ in $L^2(\Omega;\mathbb{R}^6)$ we have

$$\frac{d}{dt}\int_\Omega (Au + G_d \star u)vdx = \int_\Omega Muvdx + \int_\Omega jvdx \text{ for } v \in X_M.$$

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$$\frac{d}{dt}\int_\Omega (Au + G_d \star u)vdx = \int_\Omega Muvdx + \int_\Omega jvdx \text{ for } v \in X_M.$$

We conclude that u is a weak solution of the initial problem in $L^\infty(0, T, L^2(\Omega; \mathbb{R}^6))$ and supplemented with the above convergences provide the necessary smoothness in u.

The solution u satisfies the conservation law

$$\frac{1}{2}\int_\Omega (Au,u)dx - \int_0^t \int_\Omega j \cdot u dx ds + \int_0^t \int_\Omega G(0)u(s) \cdot u(s) dx ds + \int_0^t \int_\Omega (\dot{G} \star u(s)) ds \cdot u(s) dx ds = \frac{1}{2}\int_\Omega A u^0 \cdot u^0 dx$$

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$$+\int_0^t\int_\Omega (\dot{G}\star u(s))ds \cdot u(s)dxds = \frac{1}{2}\int_\Omega Au^0 \cdot u^0 dx$$

Proof.

The proof is based on the definition of the field

$$E(x,t) = \frac{1}{2}d(x,t)\cdot u(x,t),$$

on the property of Maxwell's operator $\int_\Omega (Mu)udx = 0$ and on $\frac{d}{dt}(Au,u) = 2(Au,\dot{u})$, $\frac{d}{dt}(G_d\star u,u) = (\dot{G}_d\star u + G_d(0)u,u) + (G_d\star u,\dot{u})$ where $(\cdot,\cdot)$ is the L_2 inner product in Ω and $\dot{f}$ is always the derivate referred to time t. □

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As a result,for any $\epsilon > 0$ there is a sequence of electromagnetic fields u^ϵ which are solutions of the evolution problem,

$$\frac{d}{dt}(A^\epsilon u^\epsilon + G_d^\epsilon \star u^\epsilon) = Mu^\epsilon - j^\epsilon, (0, T) \times \Omega$$

$$u^\epsilon(0, x) = u^{0,\epsilon}(x), \Omega$$

$$\hat{\eta}(x) \times u_1^\epsilon(t, x) = 0, (0, T) \times \partial\Omega.$$

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Purpose: The study of the asymptotic behavior of solution u^ϵ under the following assumptions:

- $u^{\epsilon,0} \to u^0$ strongly in X_M
- $J^\epsilon \to J$ strongly in $W^{1,1}(0, T; L^2(\Omega; \mathbb{R}^6))$
- A^ϵ, G_d^ϵ are periodic matrices

Let $\epsilon > 0$.We assume $Y = (0,1)^3$

$$\mathbb{Z}^3_\epsilon := m \in \mathbb{Z}^3 : \epsilon(m+Y) \subset \Omega$$

$$\Omega_\epsilon := \bigcup_{m \in \mathbb{Z}^3_\epsilon} \epsilon(m+Y)$$

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Definition 3

The periodic unfolding operator $\mathcal{T}_\epsilon : L^2(\Omega;\mathbb{R}) \to L^2(\Omega \times Y;\mathbb{R})$ is defined by

$$(\mathcal{T}_\epsilon v)(x,y) := \begin{cases} v(\epsilon[\frac{x}{\epsilon}] + \epsilon y), & x \in \Omega_\epsilon, y \in Y \\ 0, & x \in \Lambda_\epsilon, y \in Y \end{cases}$$

We can apply the unfolding operator on functions with entries functions or matrices and we have that

$$(\mathcal{T}_\epsilon A^\epsilon)(x,y) = A(y)$$

$$(\mathcal{T}_\epsilon G^\epsilon)(x,y,t) = G(y,t).$$

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From the previous definitions we deduce the following properties:

- $\mathcal{T}_\epsilon(au + bv) = a\mathcal{T}_\epsilon u + b\mathcal{T}_\epsilon v$
- $\mathcal{T}_\epsilon(uv) = (\mathcal{T}_\epsilon u)(\mathcal{T}_\epsilon v)$
- $\int_\Omega u(x)dx = \frac{1}{|Y|}\int_{\Omega\times Y}(\mathcal{T}_\epsilon u)(x,y)dydx.$

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3. *If* $u^\epsilon \in H(\mathrm{curl}, \Omega) : \|u^\epsilon\|_{H(\mathrm{curl},\Omega)} \leq c$ *for any* $\epsilon > 0$ *then there are three fields* u, v, w *which* $u \in H(\mathrm{curl}, \Omega)$*,* $v \in L^2(\Omega, H^1_{per}(Y; R))$*,* $W \in L^2(\Omega, H^1_{per}(Y; \mathbb{R}^3))$*,*$\mathrm{div}_y\, w = 0$ *and subsequence* $\{u^\epsilon\}$ *of* $\{u^\epsilon\}$ *in order to have the following convergences:*

$$\begin{aligned} u^\epsilon &\rightharpoonup u \ \ in \ \ H(\mathrm{curl}, \Omega) \\ \mathcal{T}_\epsilon u^\epsilon &\rightharpoonup u + \nabla_y v \ \ in \ \ L^2(\Omega \times Y; \mathbb{R}^3) \\ \mathcal{T}_\epsilon(\mathrm{curl}\, u^\epsilon) &\rightharpoonup \mathrm{curl}_x\, u + \mathrm{curl}_y\, w \ \ in \ \ L^2(\Omega \times Y; \mathbb{R}^3). \end{aligned}$$

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\end{aligned}
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By using the above theorem and and the conservation law for the solution u^ϵ we have the next theorem:

Theorem 5

If $u^\epsilon(x,t)$, $x \in \Omega$, $t > 0$ *the unique solution of* (P_H) *in* $W^{1,\infty}(0,T;L^2(\Omega;\mathbb{R}^6)) \bigcap L^\infty(0,T;X_M)$ *and* u, v, w *satisfying theorem 4 we obtain*

$$\begin{aligned}
&\frac{1}{2}\int_{\Omega\times Y} A(u(t)+\nabla_y v(t))(u(t)+\nabla_y v(t)) \\
&+\int_0^t\int_0^{t_1}\int_{\Omega\times Y} \dot{G}(t_1-s)(u(s)+\nabla_y v(s))(u(t_1)+\nabla_y v(t_1)) \\
&+\int_0^t\int_{\Omega\times Y} G(0)(u(s)+\nabla_y v(s))(u(s)+\nabla_y v(s)) \\
&+\int_0^t\int_{\Omega\times Y} J(s)u(s) = \frac{1}{2}\int_{\Omega\times Y} Au^0u^0.
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$$\begin{aligned}
&\frac{1}{2}\int_{\Omega\times Y} A(u(t)+\nabla_y v(t))(u(t)+\nabla_y v(t)) \\
&+ \int_0^t \int_0^{t_1} \int_{\Omega\times Y} \dot{G}(t_1-s)(u(s)+\nabla_y v(s))(u(t_1)+\nabla_y v(t_1)) \\
&+ \int_0^t \int_{\Omega\times Y} G(0)(u(s)+\nabla_y v(s))(u(s)+\nabla_y v(s)) \\
&+ \int_0^t \int_{\Omega\times Y} J(s)u(s) = \frac{1}{2}\int_{\Omega\times Y} Au^0u^0.
\end{aligned}$$

The solution u^ϵ satisfies the uniform bound

$$\|u^\epsilon\|_{L^\infty(0,T;X_M)} + \|\frac{du^\epsilon}{dt}\|_{L^\infty(0,T;L^2(\Omega;R^6))} \leq c(\|J^\epsilon\|_{W^{1,1}(0,T;L^2(\Omega,\mathbb{R}^6))} + \|u^{\epsilon,0}\|_{X_M})$$

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Theorem 6

Let $A^\epsilon \in L^\infty(\Omega;\mathbb{R}^{36})$ and $G_d^\epsilon \in W^{2,1}(0,T;L^\infty(\Omega;\mathbb{R}^{36}))$ be two matrices satisfying assumption 3 stated before.Also, the initial condition $u^{\epsilon,0}$ and the source J^ϵ satisfy assumptions 1,2 respectively then if we assume u^ϵ to be the solution if (P_H) there exists three fields u, v, w with

$$u \in W^{1,\infty}(0,T;L^2(\Omega;\mathbb{R}^6)) \bigcap L^\infty(0,T,X_M)$$

$$v \in W^{1,\infty}(0,T;L^2(\Omega;H^1_{per}(Y;\mathbb{R}^2))$$

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i) which are limits as follows

$$u^\epsilon \rightharpoonup u \ \text{ weakly}^* \text{ in } \ L^\infty(0, T; X_M)$$

$$\mathcal{T}_\epsilon u^\epsilon \rightharpoonup u + \nabla_y v \quad \text{strongly in } \ H^1(0, T; L^2(\Omega \times; \mathbb{R}^6))$$

$$\mathcal{T}_\epsilon(\operatorname{curl} u^\epsilon) \rightharpoonup \operatorname{curl}_x u + \operatorname{curl}_y w \quad \text{strongly in } \ L^2((0, T) \times \Omega \times Y; \mathbb{R}^6)$$

ii) which solve the evolution problem:

$$\frac{d}{dt}(A(y)(u(x,t)+\nabla_y v(x,y,t)))+(G_d\star(u+\nabla_y v)(t))$$
$$=M_x u(x,t)+M_y w(x,y,t)+J(x,t)$$
$$u(0)+\nabla_y v(x,y,0)=u^0$$
$$\hat{\eta}\times u_1=0$$

ii) which solve the evolution problem:

$$\begin{aligned}\frac{d}{dt}(A(y)(u(x,t) + \nabla_y v(x,y,t))) + (G_d \star (u + \nabla_y v)(t)) \\ = M_x u(x,t) + M_y w(x,y,t) + J(x,t) \\ u(0) + \nabla_y v(x,y,0) = u^0 \\ \hat{\eta} \times u_1 = 0\end{aligned}$$

Proof.

Step 1: weak convergence
Step 2: boundary condition
Step 3: strong convergence □

Thank you!